

## APPLIED MATHEMATICS - HIGHER LEVEL

Six questions to be answered. All questions carry equal marks.

Mathematics Tables may be obtained from the Superintendent.

Take the value of  $g$  to be  $9.8$  metres/second<sup>2</sup>.  $\vec{i}$  and  $\vec{j}$  are perpendicular unit vectors.

1. How may a velocity-time graph be used to find the distance travelled in a given time?

An athlete runs  $100$  m in  $12$  seconds. Starting from rest, he accelerates uniformly to a speed of  $10$  m/s, and then continues at that speed. Calculate the acceleration.

A body starting from rest travels in a straight line, first with uniform acceleration  $a$  and then with uniform deceleration  $b$ . It comes to rest when it has covered a total distance  $l$ . If the overall time for the journey is  $T$ , show that

$$T^2 = 2l\left(\frac{1}{a} + \frac{1}{b}\right)$$

2. A ship A is travelling South-West at  $10$  knots. Another ship B is travelling at  $20$  knots in a direction  $30^\circ$  North of West. Draw a diagram to show the velocity of B relative to A.

Calculate the magnitude of the relative velocity, correct to the nearest knot, and its direction correct to the nearest degree.

By how much should A increase its speed, without changing direction, so that B would appear to A to be travelling due North?

3. Explain the terms: coefficient of friction, angle of friction. What is the relationship between them?

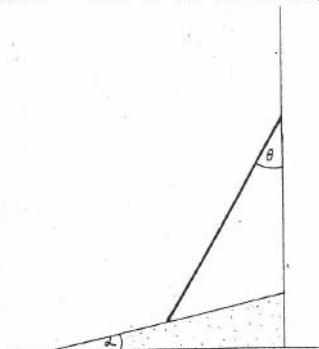
A uniform ladder of weight  $W$  rests with one end against a smooth vertical wall and the other end on rough ground which slopes away from the wall at an angle  $\alpha$  to the horizontal (see diagram). The ladder makes an angle  $\theta$  with the wall.

Show that the reaction at the wall is  $\frac{1}{2}W \tan \theta$ .

If the ladder is on the point of slipping prove that

$$\tan \theta = 2 \tan (\lambda - \alpha)$$

where  $\lambda$  is the angle of friction.



4. (a) A see-saw consists of a uniform plank freely pivoted at its mid-point on a support of vertical height  $h$ . It carries a mass  $m$  at one end and a mass  $2m$  at the other. The see-saw is released from rest with the mass  $2m$  at its highest point. Find, in terms of  $h$ , the velocity with which the see-saw strikes the ground.

(b) State the Principle of Archimedes.

A uniform rod of length  $l$  and relative density  $s$  is freely hinged to the base of a tank containing a liquid to a depth  $h$ . The relative density of the liquid is  $k$ . As a result, the rod is inclined but not fully submerged. Derive an expression for the angle the rod makes with the horizontal.

5. A plane is inclined at an angle  $\alpha$  to the horizontal. A particle is projected up the plane with a velocity  $u$  at an angle  $\theta$  to the plane. The plane of projection is vertical and contains the line of greatest slope.

(i) Show that the time of flight is  $2u \sin \theta / g \cos \alpha$ .

(ii) Prove that the range up the plane is a maximum when  $\theta = \frac{1}{2}(\frac{\pi}{2} - \alpha)$ .

(iii) Prove that the particle will strike the plane horizontally if  $\tan \theta = \sin \alpha \cos \alpha / (2 - \cos^2 \alpha)$ .

6. (a) A particle, moving at constant speed, is describing a horizontal circle on the inside surface of a smooth sphere of radius  $r$ . The centre of the circle is a distance  $\frac{1}{2}r$  below the centre of the sphere. Prove that the speed of the particle is  $\frac{1}{2}\sqrt{6gr}$ .
- (b) A conical pendulum consists of a light elastic string with a mass  $m$  attached to it which is rotating with uniform angular velocity  $\omega$ . The natural length of the string is  $l$  and its elastic constant is  $k$ , i.e. a force  $k$  produces unit extension. The extended length of the string is  $l'$  and it makes an angle  $\theta$  with the vertical. Prove that  $k(l' - l) = ml'\omega^2$  and that

$$\cos \theta = \frac{g}{l\omega^2} - \frac{mg}{kl}$$

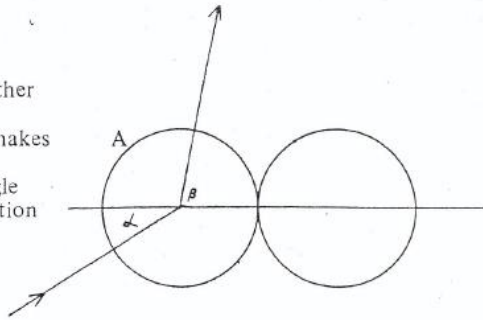
7. (a) Two smooth spheres of masses  $m$  and  $2m$  collide directly when moving in opposite directions with speeds  $u$  and  $v$ , respectively. The sphere of mass  $2m$  is brought to rest by the impact. Prove that

$$e = \frac{2v - u}{u + v}$$

where  $e$  is the coefficient of restitution.

- (b) A smooth sphere A collides obliquely with another smooth sphere of equal mass which is at rest. Before impact the direction of motion of A makes an angle  $\alpha$  with the line of centres at impact (see diagram). After impact it makes an angle  $\beta$  with that line. If the coefficient of restitution is  $\frac{1}{2}$ , prove that

$$\tan \beta = 4 \tan \alpha$$



8. (a) Define simple harmonic motion. Using the usual notation, show that the equation  $\dot{v}^2 = \omega^2 (a^2 - x^2)$  represents simple harmonic motion. A body is moving with simple harmonic motion, of amplitude 5 m. When it is 4 m from the mid-point of its path its speed is 6 m/s. Find its speed when it is 2.5 m from the mid-point.
- (b) A block rests on a rough platform which moves to and fro horizontally with simple harmonic motion. The amplitude is 0.75 m and 20 complete oscillations occur per minute. If the block remains at rest relative to the platform throughout the motion, find the least possible value the coefficient of friction can have.

9. State the theorem of parallel axes.

Prove that the moment of inertia of a uniform rod of mass  $m$  and length  $2l$  about a perpendicular axis through one of its ends is  $4ml^2/3$ .

A uniform rod of length  $6l$  is attached to the rim of a uniform disc of diameter  $2l$ . The rod is collinear with a diameter of the disc (see diagram). The disc and the rod are both of mass  $m$ . Calculate the moment of inertia of the compound body about a perpendicular axis through the end A.

If the compound body makes small oscillations in a vertical plane about a horizontal axis through A, show that the periodic time is  $2\pi\sqrt{123l/20g}$ .



10. (a) Solve the differential equation

$$x \frac{dy}{dx} = -y$$

Hence or otherwise solve

$$x \frac{d^2y}{dx^2} = -\frac{dy}{dx}$$

where  $y = 0$  when  $x = 1$  and  $y = 3$  when  $x = e$ .

- (b) A body is moving in a straight line subject to a deceleration which is equal to  $v^2/10$ , where  $v$  is the velocity. The initial velocity is 5 m/s. In how many seconds will the velocity of the body be 2 m/s and how far will it travel in that time?