

## APPLIED MATHEMATICS - HIGHER LEVEL

Six questions to be answered. All questions carry equal marks.

Mathematics Tables may be obtained from the Superintendent.

Take the value of  $g$  to be  $9.8$  metres/second<sup>2</sup>.  $\vec{i}$  and  $\vec{j}$  are perpendicular unit vectors.

1. A driver starts from rest at P and travels with a uniform acceleration of  $a$  m/s<sup>2</sup> for  $T$  seconds. He continues with uniform velocity for  $3T$  seconds, and then decelerates uniformly to rest at Q in a further  $2T$  seconds. Express the distance PQ in terms of  $a$  and  $T$ .

Another driver can accelerate at  $2a$  m/s<sup>2</sup> and can decelerate at  $4a$  m/s<sup>2</sup>. Find, in terms of  $T$ , the least time in which this driver can cover the distance PQ from rest to rest

- subject to a speed limit of  $3aT$  m/s,
- subject to a speed limit of  $5aT$  m/s.

2. A plane is inclined at an angle  $\alpha$  to the horizontal. A particle is projected up the plane with initial velocity  $u$  at an angle  $\theta$  to the plane. The plane of projection is vertical and contains the line of greatest slope. Write down the displacement and velocity of the particle parallel and perpendicular to the plane at time  $t$ .

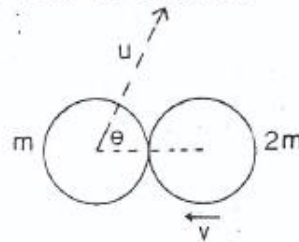
Show that the time taken by the particle to reach its maximum perpendicular height above the plane is half the time of flight up the plane.

When the particle is at its maximum perpendicular height above the plane, the distance travelled parallel to the plane is  $\frac{2}{3}$  of the range up the plane. Show that in that case

$$\tan \theta \tan \alpha = \frac{2}{3}.$$

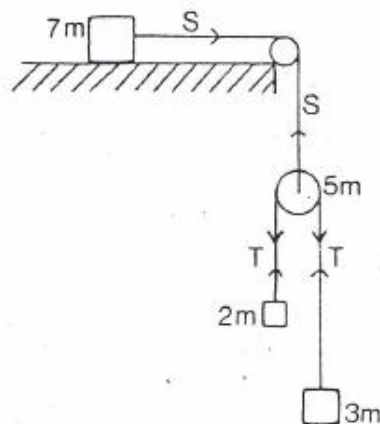
3. Two vectors  $a\vec{i} + b\vec{j}$  and  $c\vec{i} + d\vec{j}$  are at right angles. Write down the condition satisfied by the scalars  $a, b, c, d$ .

Two smooth spheres of masses  $m$  and  $2m$  and velocities  $u$  and  $v$ , respectively, collide as shown in the diagram, where  $\cos \theta = \frac{3}{7}$ . The sphere of mass  $m$  is deflected through an angle of  $90^\circ$  by the collision. If the coefficient of restitution is  $\frac{1}{2}$ , show that  $v = \frac{11u}{7}$ .



Find the direction of motion of the other sphere after the collision.

4. A body of mass  $7m$  lies on a smooth horizontal table. It is connected by means of a light string passing over a smooth light pulley at the edge of the table, to a second smooth pulley of mass  $5m$  hanging freely. Over this second pulley passes another light string carrying masses of  $2m$  and  $3m$  (see diagram).



Show in separate diagrams the forces acting on each of the masses. Write down the equations of motion involving the tensions  $T$  and  $S$  in the strings, the common acceleration  $f$  of the  $7m$  and  $5m$  masses and the common acceleration  $a$  of the  $3m$  and  $2m$  masses relative to the  $5m$  mass.

Show that  $f = \frac{7}{13}g$ .

5. Two uniform rods AB and BC of equal length and of masses  $5$  kg and  $3$  kg, respectively, are freely hinged at B. AB and BC are in a vertical plane and the ends A and C are on a rough horizontal plane. The coefficient of friction between each rod and the plane is the same. Find the normal reactions at A and C.

The angle ABC is increased until one of the rods begins to slip. Show that slipping will first occur at C rather than at A.

Find the least value of the coefficient of friction if slipping has not occurred before  $\angle ABC = 90^\circ$ .

6. Two small smooth rings A and B, each of mass  $m$  are threaded on a fixed smooth horizontal wire. They are connected by means of two light inextensible strings AC and BC, each of length  $2$  metres, to a particle of mass  $2m$  hanging freely at C. A, B, C are in the same vertical plane. The system is released from rest with the angle  $\angle BAC = 30^\circ$ .

If A travels a horizontal distance  $x$  while C falls a vertical distance  $y$ , show from geometry that  $x^2 + y^2 - 2\sqrt{3}x + 2y = 0$ .

By differentiating find  $\dot{y}/\dot{x}$  in terms of  $x$  and  $y$ , where  $\dot{x}$  means  $\frac{dx}{dt}$ , and using the conservation of energy find  $\dot{x}$  in terms of  $y$ .

Show that the velocity of A is  $\sqrt{g(\sqrt{2}-1)}$  when  $\angle BAC = 45^\circ$ .

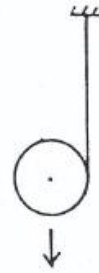
7. Prove that the moment of inertia of a uniform circular disc about a perpendicular axis through its centre is  $\frac{1}{2} m r^2$ , where  $r$  is the radius of the disc and  $m$  is its mass.

A light string is wound around the rim of a uniform disc of radius  $r$  and mass  $m$ . One end of the string is attached to the rim of the disc and the other end is attached to a fixed point above the disc, with the plane of the disc vertical (see diagram). When the disc is released from rest it falls vertically and the string unwinds.

If the disc falls a distance  $x$  while it turns through an angle  $\theta$ , show that  $x = r\theta$  and deduce that  $\dot{x} = r\dot{\omega}$ , where  $\omega$  is the angular velocity of the disc.

( $\dot{x}$  means  $\frac{dx}{dt}$ ,  $\ddot{x}$  means  $\frac{d^2x}{dt^2}$ )

Using the principle of angular momentum, find the tension in the string and the vertical acceleration of the disc.



8. Solve the following differential equations:

(i)  $\frac{dy}{dx} = \sqrt{1 - y^2}$  if  $y = 0$  when  $x = 1$

(ii)  $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$  if  $\frac{dy}{dx} = 1$  and  $x = \frac{1}{2}$  when  $y = 1$ .

A particle of mass  $m$  is acted on by a force  $\frac{2m}{x^5}$  directed away from a fixed point O, where  $x$  is the distance of the particle from O. The particle starts from rest at a distance  $d$  from O. Show that the velocity of the particle tends to a limit  $1/d^2$ .

9. (a) Two particles A and B are moving along two perpendicular lines towards a point O with constant velocities of 1.2 m/s and 1.6 m/s respectively. When A is 12 metres from O, B is 20 metres from O. Find the distance between them when they are nearest to each other.

(b) State the Principle of Archimedes.

A uniform circular cylinder of height  $h$  and relative density  $s$  floats with its axis vertical in a liquid of relative density  $w$ . Find the length of the axis of the cylinder immersed.

The cylinder is depressed vertically a further small distance  $x$  and released. Show that it will perform simple harmonic motion, and find the period.

10. (a) A portion in the shape of an equilateral triangle is removed from a circular lamina of radius  $r$ . A vertex of the triangle was at the centre of the lamina and the sides of the triangle are of length  $r$ . Find the position of the centre of gravity of the remainder.

(b) A train of mass 300 tonnes is maintaining a steady speed of 10 m/s up an incline of 1 in 120 against frictional forces amounting to 40 kN. Calculate the power at which the engine is working. (1 tonne = 1000 kg)

(c) A corner on a level track has a radius of 100 m. Calculate the maximum speed at which a cyclist could take the corner if the coefficient of friction were  $\frac{1}{4}$ .