

Chapter 12 Exercise 12A

- Q. 1.**
- (i) $\log_e 4 + \log_e 3 = \log_e 12$
- (ii) $\log_e 6 - \log_e 7 = \log_e \frac{6}{7}$
- (iii) $2 \log_e 3 + 3 \log_e 2 = \log_e 3^2 + \log_e 2^3$
 $= \log_e 9 + \log_e 8$
 $= \log_e 72$
- (iv) $5 \log_e 2 - 2 \log_e 5 = \log_e 2^5 - \log_e 5^2$
 $= \log_e 32 - \log_e 25$
 $= \log_e \frac{32}{25}$
- (v) $\frac{1}{2} \log_e 4 + \frac{1}{3} \log_e 27$
 $= \log_e 4^{\frac{1}{2}} + \log_e 27^{\frac{1}{3}}$
 $= \log_e 2 + \log_e 3$
 $= \log_e 6$
- (vi) $\frac{1}{2} \log_e 64 - \frac{1}{3} \log_e 64$
 $= \log_e 8 - \log_e 4$
 $= \log_e 2$
- (vii) $2 \log_e 10 + \log_e 6 - 3 \log_e 4$
 $= \log_e 100 + \log_e 6 - \log_e 64$
 $= \log_e \left(\frac{100 \times 6}{64} \right)$
 $= \log_e \left(\frac{75}{8} \right)$
- (viii) $\log_e x^2 + \log_e x = \log_e x^3$
- (ix) $\frac{1}{2} \log_e x - \log_e 7 + \log_e 2$
 $= \log_e \sqrt{x} - \log_e 7 + \log_e 2$
 $= \log_e \frac{2\sqrt{x}}{7}$
- Q. 2.**
- (i) $e^x = e$
 $\Rightarrow x = 1$
- (ii) $x = e^2$
- (iii) $e^x = \frac{1}{e}$
 $\Rightarrow x = -1$
- (iv) $e^x = \sqrt[3]{e}$
 $\Rightarrow x = \frac{1}{3}$
- (v) $e^{\log_e x} = 8$
 $\Rightarrow x = 8$ (since e and \log_e are inverse functions).
- (vi) $\log_e(e^4) = x$
 $\Rightarrow e^x = e^4$
 $\Rightarrow x = 4$
- (vii) $e^{\log_e 2} = x$
 $\Rightarrow 2 = x$
- (viii) $e^{\frac{1}{2} \log_e x} = 7$
 $\Rightarrow e^{\log_e \sqrt{x}} = 7$
 $\Rightarrow \sqrt{x} = 7$
 $\Rightarrow x = 49$
- (ix) $e^{2 \log_e x} = 9$
 $\Rightarrow e^{\log_e x^2} = 9$
 $\Rightarrow x^2 = 9$
 $\Rightarrow x = +3$
 (-3 is not possible since $\log_e(-3)$ doesn't exist).
- (x) $e^{3 \log_e 4} = x$
 $\Rightarrow e^{\log_e 64} = x$
 $\Rightarrow x = 64$
- (xi) $\log_e x = 1 + \log_e 2$
 $\Rightarrow \log_e x - \log_e 2 = 1$
 $\Rightarrow \log_e \frac{x}{2} = 1$
 $\Rightarrow \frac{x}{2} = e^1$
 $\Rightarrow x = 2e$
- (xii) $\log_e x = 3 - \log_e 5$
 $\Rightarrow \log_e x + \log_e 5 = 3$
 $\Rightarrow \log_e 5x = 3$
 $\Rightarrow 5x = e^3$
 $\Rightarrow x = \frac{1}{5}e^3$
- (xiii) $2 \log_e x = 1 - \log_e 3$
 $\Rightarrow \log_e x^2 = 1 - \log_e 3$
 $\Rightarrow \log_e x^2 + \log_e 3 = 1$
 $\Rightarrow \log_e 3x^2 = 1$
 $\Rightarrow 3x^2 = e$
 $\Rightarrow x = \sqrt{\frac{e}{3}}$

$$\begin{aligned}
 \text{(xiv)} \quad \frac{1}{2} \log_e x + 1 &= \frac{3}{2} \log_e 3 \\
 &\Rightarrow \log_e x + 2 = 3 \log_e 3 \\
 &\Rightarrow \log_e x - 3 \log_e 3 = -2 \\
 &\Rightarrow \log_e \left(\frac{x}{27} \right) = -2 \\
 &\Rightarrow \frac{x}{27} = e^{-2} \\
 &\Rightarrow x = 27e^{-2}
 \end{aligned}$$

Exercise 12B

- Q. 1.** (i) $\int x^4 dx = \frac{x^5}{5} + c$
 (ii) $\int 3x^4 dx = \frac{3x^5}{5} + c$
- Q. 2.** (i) $\int \cos x dx = \sin x + c$
 (ii) $\int \cos 3x dx = \frac{\sin 3x}{3} + c$
- Q. 3.** (i) $\int \sin x dx = -\cos x + c$
 (ii) $\int \sin 4x dx = -\frac{\cos 4x}{4} + c$
- Q. 4.** (i) $\int \frac{1}{x} dx = \ln x + c$
 (ii) $\int \frac{1}{2x+3} dx = \frac{\ln(2x+3)}{2} + c$
- Q. 5.** (i) $\int e^x dx = e^x + c$
 (ii) $\int e^{8x} dx = \frac{e^{8x}}{8} + c$
- Q. 6.** (i) $\int \frac{1}{\sqrt{49-x^2}} dx = \sin^{-1} \frac{x}{7} + c$
 (ii) $\int \frac{dx}{\sqrt{100-x^2}} = \sin^{-1} \frac{x}{10} + c$
- Q. 7.** (i) $\int \frac{1}{x^2+25} dx = \frac{1}{5} \tan^{-1} \frac{x}{5} + c$
 (ii) $\int \frac{dx}{x^2+625} = \int \frac{dx}{x^2+(25)^2}$
 $= \frac{1}{25} \tan^{-1} \frac{x}{25} + c$
- Q. 8.** (i) $\int \frac{x+1}{x} dx = \int \left(1 + \frac{1}{x} \right) dx$
 $= x + \ln x + c$
 (ii) $\int \frac{1}{x+1} dx = \ln(x+1) + c$

- Q. 9.** (i) $\int (2x+1) dx = x^2 + x + c$
 (ii) $\int (2x+1)^2 dx = \int (4x^2 + 4x + 1) dx$
 $= \frac{4x^3}{3} + 2x^2 + x + c$
- Q. 10.** (i) $\int \frac{1}{5x+1} dx = \frac{\ln(5x+1)}{5} + c$
 (ii) $\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{x^{\frac{1}{2}}} dx$
 $= \int x^{-\frac{1}{2}} dx$
 $= 2x^{\frac{1}{2}} + c$
 $= 2\sqrt{x} + c$

Exercise 12C

- Q. 1.** (i) $\int_1^3 4x dx = 2[x^2]_1^3 = 2[9-1] = 16$
 (ii) $\int_0^1 x^2 dx = \frac{1}{3}[x^3]_0^1 = \frac{1}{3}$
 (iii) $\int_1^2 (2x+1) dx = [x^2 + x]_1^2$
 $= (4+2) - (1+1) = 4$
 (iv) $\int_1^4 (2-x^2) dx = \left[2x - \frac{2x^3}{3} \right]_1^4$
 $= 2 \left[x - \frac{x\sqrt{x}}{3} \right]_1^4$
 $= 2 \left[\left(4 - \frac{4\sqrt{4}}{3} \right) - \left(1 - \frac{1\sqrt{1}}{3} \right) \right]$
 $= \frac{4}{3}$
 (v) $\int_2^6 \frac{1}{x} dx = [\ln x]_2^6 = \ln 6 - \ln 2$
 $= \ln \frac{6}{2} = \ln 3$
 (vi) $\int_{-1}^1 x^3 dx = \frac{1}{4}[x^4]_{-1}^1$
 $= \frac{1}{4}[(1)^4 - (-1)^4] = 0$
- Q. 2.** $\int_0^{\ln 4} e^x dx = [e^x]_0^{\ln 4}$
 $= e^{\ln 4} - e^0$
 $= 4 - 1$
 $= 3$
- Q. 3.** $\int_0^{\frac{\pi}{2}} \cos x dx = [\sin x]_0^{\frac{\pi}{2}}$
 $= \sin \frac{\pi}{2} - \sin 0$
 $= 1$

$$\begin{aligned} \text{Q. 4. } \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \, dx &= -\left[\cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= -\left[\cos \frac{\pi}{2} - \cos \frac{\pi}{4}\right] \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{Q. 5. } \int_{\ln 2}^{\ln 5} e^x \, dx &= \left[e^x\right]_{\ln 2}^{\ln 5} \\ &= e^{\ln 5} - e^{\ln 2} \\ &= 5 - 2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Q. 6. } 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x \, dx &= 2\left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= 2\left[\sin \frac{\pi}{3} - \sin \frac{\pi}{6}\right] \\ &= \sqrt{3} - 1 \end{aligned}$$

$$\begin{aligned} \text{Q. 7. } 4 \int_0^{\frac{\pi}{3}} \sin x \, dx &= -4\left[\cos x\right]_0^{\frac{\pi}{3}} \\ &= -4\left[\cos \frac{\pi}{3} - \cos 0\right] \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{Q. 8. } \int_0^{\frac{\pi}{4}} (\cos x + \sin x) \, dx &= \left[\sin x - \cos x\right]_0^{\frac{\pi}{4}} \\ &= \left(\sin \frac{\pi}{4} - \cos \frac{\pi}{4}\right) - (\sin 0 - \cos 0) = 1 \end{aligned}$$

$$\begin{aligned} \text{Q. 9. } \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x) \, dx &= \left[\sin x + \cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \left(\sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) - \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) \\ &= 1 - \sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Q. 10. } \int_2^4 \frac{dx}{\sqrt{16-x^2}} &= \left[\sin^{-1} \frac{x}{4}\right]_2^4 \\ &= \sin^{-1} 1 - \sin^{-1} \frac{1}{2} \\ &= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{Q. 11. } \int_0^{\ln 2} e^{3x} \, dx &= \frac{1}{3} \left[e^{3x}\right]_0^{\ln 2} \\ &= \frac{1}{3} [e^{3 \ln 2} - e^0] \\ &= \frac{1}{3} [e^{\ln(2)^3} - 1] \\ &= \frac{1}{3} [e^{\ln 8} - 1] = \frac{1}{3} [8 - 1] = \frac{7}{3} \end{aligned}$$

$$\begin{aligned} \text{Q. 12. } \int_0^{\frac{\pi}{16}} \cos 4x \, dx &= \frac{1}{4} \left[\sin 4x\right]_0^{\frac{\pi}{16}} \\ &= \frac{1}{4} \left[\sin\left(\frac{\pi}{4}\right) - \sin 0\right] \\ &= \frac{1}{4\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{Q. 13. } \int_0^{\frac{\pi}{6}} \sin 2x \, dx &= -\frac{1}{2} \left[\cos 2x\right]_0^{\frac{\pi}{6}} \\ &= -\frac{1}{2} \left[\cos \frac{\pi}{3} - \cos 0\right] \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{Q. 14. } \int_0^{\frac{\pi}{6}} \cos 3x \, dx &= +\frac{1}{3} \left[\sin 3x\right]_0^{\frac{\pi}{6}} \\ &= +\frac{1}{3} \left[\sin \frac{\pi}{2} - \sin 0\right] \\ &= +\frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{Q. 15. } \int_0^{\frac{\pi}{12}} \sin 6x \, dx &= -\frac{1}{6} \left[\cos 6x\right]_0^{\frac{\pi}{12}} \\ &= -\frac{1}{6} \left[\cos \frac{\pi}{2} - \cos 0\right] \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \text{Q. 16. (i) } \int_0^2 \frac{dx}{x^2+4} &= \frac{1}{2} \left[\tan^{-1} \frac{x}{2}\right]_0^2 \\ &= \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0] \\ &= \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} \text{(ii) } \int_0^{\frac{\pi}{2}} \cos^2 x \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \left[\frac{\pi}{2} + \frac{\sin \pi}{2}\right] \\ &= \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{Q. 17. (i)} \quad & \int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} \\ &= \left[\sin^{-1} \frac{x}{2} \right]_1^{\sqrt{3}} \\ &= \left[\sin^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} \frac{1}{2} \right] \\ &= \frac{\pi}{3} - \frac{\pi}{6} \\ &= \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \int_0^{\frac{\pi}{3}} \sin^2 x \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sin \frac{2\pi}{3}}{2} \right] \\ &= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{8} \end{aligned}$$

$$\begin{aligned} \text{Q. 18. (i)} \quad & \int_0^9 \frac{dx}{x^2 + 81} \\ &= \frac{1}{9} \left[\tan^{-1} \frac{x}{9} \right]_0^9 \\ &= \frac{1}{9} [\tan^{-1} 1 - \tan^{-1} 0] \\ &= \frac{\pi}{36} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \int_0^{\frac{\pi}{4}} \cos^2 2x \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 + \cos 4x) \, dx \\ &= \frac{1}{2} \left[x + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} \left[\frac{\pi}{4} + 0 \right] \\ &= \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} \text{Q. 19. (i)} \quad & \int_0^1 \frac{dx}{\sqrt{2-x^2}} \\ &= \int_0^1 \frac{dx}{\sqrt{(\sqrt{2})^2 - x^2}} \\ &= \left[\sin^{-1} \left(\frac{x}{\sqrt{2}} \right) \right]_0^1 \\ &= \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) \\ &= \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \int_0^{\frac{\pi}{8}} \sin^2(2x) \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{8}} (1 - \cos 4x) \, dx \\ &= \frac{1}{2} \left[x - \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{8}} \\ &= \frac{1}{2} \left[\frac{\pi}{8} - \frac{1}{4} \right] \\ &= \frac{1}{8} \left[\frac{\pi}{2} - 1 \right] \end{aligned}$$

$$\begin{aligned} \text{Q. 20. (i)} \quad & \int_1^{\sqrt{3}} \frac{dx}{x^2 + 3} \\ &= \int_1^{\sqrt{3}} \frac{dx}{x^2 + (\sqrt{3})^2} \\ &= \frac{1}{\sqrt{3}} \left[\tan^{-1} \frac{x}{\sqrt{3}} \right]_1^{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \left[\tan^{-1} 1 - \tan^{-1} \frac{1}{\sqrt{3}} \right] \\ &= \frac{1}{\sqrt{3}} \left[\frac{\pi}{4} - \frac{\pi}{6} \right] \\ &= \frac{\pi}{12\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \int_0^{\sqrt{2}} \frac{dx}{\sqrt{8-x^2}} \\ &= \int_0^{\sqrt{2}} \frac{dx}{\sqrt{(\sqrt{8})^2 - x^2}} \\ &= \left[\sin^{-1} \frac{x}{\sqrt{8}} \right]_0^{\sqrt{2}} \\ &= \sin^{-1} \frac{\sqrt{2}}{8} \\ &= \sin^{-1} \frac{1}{2} \\ &= \frac{\pi}{6} \end{aligned}$$

Exercise 12D

$$\begin{aligned} \text{Q. 1.} \quad & \frac{dy}{dx} = 6y \\ & \Rightarrow \int \frac{dy}{y} = 6 \int dx \\ & \Rightarrow \ln y = 6x + c \\ & \Rightarrow y = e^{6x+c} \end{aligned}$$

$$\begin{aligned} \text{Q. 2.} \quad & \frac{dy}{dx} = 2xy \\ & \Rightarrow \int \frac{dy}{y} = 2 \int x \, dx \\ & \Rightarrow \ln y = x^2 + c \\ & \Rightarrow y = e^{x^2+c} \end{aligned}$$

Q. 3. $\frac{dy}{dx} = \cos x$
 $\Rightarrow \int dy = \int \cos x \, dx$
 $\Rightarrow y = \sin x + c$

Q. 4. $\frac{dy}{dx} = \frac{4x^3}{y}$
 $\Rightarrow \int y \, dy = 4 \int x^3 \, dx$
 $\Rightarrow \frac{y^2}{2} = x^4 + c$
 $\Rightarrow y^2 = 2(x^4 + c)$
 $\Rightarrow y = \pm \sqrt{2(x^4 + c)}$

Q. 5. $\frac{dy}{dx} = 3x^2y$
 $\Rightarrow \int \frac{dy}{y} = 3 \int x^2 \, dx$
 $\Rightarrow \ln y = x^3 + c$
 $\Rightarrow y = e^{(x^3 + c)}$

Q. 6. $\frac{dy}{dx} + \frac{\sin 2x}{y} = 0$
 $\Rightarrow \int \frac{dy}{y} = -\frac{\sin 2x}{y}$
 $\Rightarrow \int y \, dy = -\int \sin 2x \, dx$
 $\Rightarrow \frac{y^2}{2} = \frac{\cos 2x}{2} + c$
 $\Rightarrow y^2 = \cos 2x + 2c$
 $\Rightarrow y = \pm \sqrt{\cos 2x + 2c}$

Q. 7. $\frac{dy}{dx} = 2x(y^2 + 1)$
 $\Rightarrow \int \frac{dy}{y^2 + 1} = 2 \int x \, dx$
 $\Rightarrow \tan^{-1} y = x^2 + c$
 $\Rightarrow y = \tan(x^2 + c)$

Q. 8. $\frac{dy}{dx} = 2\sqrt{1 - y^2}$
 $\Rightarrow \int \frac{dy}{\sqrt{1 - y^2}} = 2 \int dx$
 $\Rightarrow \sin^{-1} y = 2x + c$
 $\Rightarrow y = \sin(2x + c)$

Exercise 12E

Q. 1. $\frac{dy}{dx} = 3y$
 $\Rightarrow \frac{1}{y} \, dy = 3 \, dx$
 $\Rightarrow \int_1^y \frac{1}{y} \, dy = \int_0^x 3 \, dx$
 $\Rightarrow \log_e y \Big|_1^y = 3x \Big|_0^x$
 $\Rightarrow \log_e y - \log_e 1 = 3x - 3(0)$
 $\Rightarrow \log_e y = 3x$
 $\Rightarrow y = e^{3x}$

Q. 2. $\frac{dy}{dx} = 5y$
 $\Rightarrow \frac{1}{y} \, dy = 5 \, dx$
 $\Rightarrow \int_2^y \frac{1}{y} \, dy = \int_0^x 5 \, dx$
 $\Rightarrow \log_e y \Big|_2^y = 5x \Big|_0^x$
 $\Rightarrow \log_e y - \log_e 2 = 5x - 5(0)$
 $\Rightarrow \log_e \frac{y}{2} = 5x$
 $\Rightarrow \frac{y}{2} = e^{5x}$
 $\Rightarrow y = 2e^{5x}$

Q. 3. $\frac{dy}{dx} = 2x\sqrt{1 - y^2}$
 $\Rightarrow \frac{1}{\sqrt{1 - y^2}} \, dy = 2x \, dx$
 $\Rightarrow \int_1^y \frac{1}{\sqrt{1 - y^2}} \, dy = \int_0^x 2x \, dx$
 $\Rightarrow \sin^{-1} y \Big|_1^y = x^2 \Big|_0^x$
 $\Rightarrow \sin^{-1} y - \sin^{-1} 1 = x^2 - 0^2$
 $\Rightarrow \sin^{-1} y - \frac{\pi}{2} = x^2$
 $\Rightarrow \sin^{-1} y = x^2 + \frac{\pi}{2}$
 $\Rightarrow y = \sin \left(x^2 + \frac{\pi}{2} \right)$

Q. 4. $\frac{dx}{dt} = tx$
 $\Rightarrow \frac{1}{x} \, dx = t \, dt$
 $\Rightarrow \int_{\sqrt{e}}^x \frac{1}{x} \, dx = \int_1^t t \, dt$
 $\Rightarrow \log_e x \Big|_{\sqrt{e}}^x = \frac{t^2}{2} \Big|_1^t$
 $\Rightarrow \log_e x - \log_e \sqrt{e} = \frac{t^2}{2} - \frac{1}{2}$
 $\Rightarrow \log_e x - \frac{1}{2} \log_e e = \frac{t^2 - 1}{2}$

$$\Rightarrow 2\log_e x - 1 = t^2 - 1$$

$$\Rightarrow \log_e x = \frac{t^2}{2}$$

$$\Rightarrow x = e^{\frac{t^2}{2}}$$

Q. 5. $\frac{dy}{dx} = \frac{y}{x}$

$$\Rightarrow \frac{1}{y} dy = \frac{1}{x} dx$$

$$\Rightarrow \int_3^y \frac{1}{y} dy = \int_1^x \frac{1}{x} dx$$

$$\Rightarrow \log_e y \Big|_3^y = \log_e x \Big|_1^x$$

$$\Rightarrow \log_e y - \log_e 3 = \log_e x - \log_e 1$$

$$\Rightarrow \log_e \frac{y}{3} = \log_e x$$

$$\Rightarrow \frac{y}{3} = x \quad \dots \text{let } y = 21$$

$$\Rightarrow x = 7$$

Q. 6. $\frac{ds}{dt} + \frac{\sin t}{s} = 0$

$$\Rightarrow \frac{ds}{dt} = \frac{-\sin t}{s}$$

$$\Rightarrow s ds = -\sin t dt$$

$$\Rightarrow \int_{\sqrt{2}}^s s ds = \int_{\frac{\pi}{3}}^t (-\sin t) dt$$

$$\Rightarrow \frac{s^2}{2} \Big|_{\sqrt{2}}^s = \cos t \Big|_{\frac{\pi}{3}}^t$$

$$\Rightarrow \frac{s^2}{2} - 1 = \cos t - \cos \frac{\pi}{3}$$

$$\Rightarrow \frac{s^2}{2} - 1 = \cos t - \frac{1}{2}$$

$$\Rightarrow s^2 - 2 = 2 \cos t - 1$$

$$\Rightarrow s^2 = 2 \cos t + 1$$

$$\Rightarrow s = \sqrt{2 \cos t + 1}$$

Q. 7. $\frac{dy}{dx} - 4x^3y = 0$

$$\Rightarrow \frac{dy}{dx} = 4x^3y$$

$$\Rightarrow \frac{1}{y} dy = 4x^3 dx$$

$$\Rightarrow \int_3^y \frac{1}{y} dy = \int_0^x 4x^3 dx$$

$$\Rightarrow \log_e y \Big|_3^y = x^4 \Big|_0^x$$

$$\Rightarrow \log_e y - \log_e 3 = x^4 - 0^4$$

$$\Rightarrow \log_e \frac{y}{3} = x^4$$

$$\Rightarrow \frac{y}{3} = e^{x^4}$$

$$\Rightarrow y = 3e^{x^4}$$

Q. 8. $\frac{y}{x^2} \frac{dy}{dx} = 1$

$$\Rightarrow y dy = x^2 dx$$

$$\Rightarrow \int_0^y y dy = \int_1^x x^2 dx$$

$$\Rightarrow \frac{y^2}{2} \Big|_0^y = \frac{x^3}{3} \Big|_1^x$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} - \frac{1}{3}$$

$$\Rightarrow 3y^2 = 2x^3 - 2$$

$$\Rightarrow y^2 = \frac{2}{3}(x^3 - 1)$$

$$\Rightarrow y = \sqrt{\frac{2}{3}(x^3 - 1)}$$

Q. 9. $\frac{dy}{dx} - \frac{\cos x}{y} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{y}$$

$$\Rightarrow y dy = \cos x dx$$

$$\Rightarrow \int_1^y y dy = \int_{\frac{\pi}{2}}^x \cos x dx$$

$$\Rightarrow \frac{y^2}{2} \Big|_1^y = \sin x \Big|_{\frac{\pi}{2}}^x$$

$$\Rightarrow \frac{y^2}{2} - \frac{1}{2} = \sin x - \sin \frac{\pi}{2}$$

$$\Rightarrow \frac{y^2}{2} - \frac{1}{2} = \sin x - 1$$

$$\Rightarrow y^2 - 1 = 2 \sin x - 2$$

$$\Rightarrow y^2 = 2 \sin x - 1$$

$$\Rightarrow y = \sqrt{2 \sin x - 1}$$

Q. 10. $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y+1}{x}$$

$$\Rightarrow \frac{1}{y+1} dy = \frac{1}{x} dx$$

$$\Rightarrow \int_0^y \frac{1}{y+1} dy = \int_4^x \frac{1}{x} dx$$

$$\Rightarrow \log_e |y+1| \Big|_0^y = \log_e x \Big|_4^x$$

$$\Rightarrow \log_e |y+1| - \log_e 1 = \log_e x - \log_e 4$$

$$\Rightarrow \log_e |y+1| = \log_e \frac{x}{4}$$

$$\Rightarrow y+1 = \frac{x}{4}$$

$$\Rightarrow y = \frac{x}{4} - 1$$

Q. 11. $\frac{dy}{dx} - y^2 \sin x = 0$

$$\Rightarrow \frac{dy}{dx} = y^2 \sin x$$

$$\Rightarrow \frac{1}{y^2} dy = \sin x dx$$

$$\Rightarrow \int_1^y \frac{1}{y^2} dy = \int_\pi^x \sin x dx$$

$$\Rightarrow \left[-\frac{1}{y} \right]_1^y = -\cos x \Big|_\pi^x$$

$$\Rightarrow \left[\frac{1}{y} \right]_1^y = \cos x \Big|_\pi^x$$

$$\Rightarrow \frac{1}{y} - 1 = \cos x - \cos \pi$$

$$\Rightarrow \frac{1}{y} - 1 = \cos x - (-1)$$

$$\Rightarrow \frac{1}{y} = \cos x + 2$$

$$\Rightarrow y = \frac{1}{\cos x + 2}$$

Q. 12. $xy \frac{dy}{dx} = y^2$

$$\Rightarrow \frac{1}{y} dy = \frac{1}{x} dx$$

$$\Rightarrow \int_2^y \frac{1}{y} dy = \int_1^x \frac{1}{x} dx$$

$$\Rightarrow \log_e y \Big|_2^y = \log_e x \Big|_1^x$$

$$\Rightarrow \log_e y - \log_e 2 = \log_e x - \log_e 1$$

$$\Rightarrow \log_e \frac{y}{2} = \log_e x$$

$$\Rightarrow \frac{y}{2} = x$$

$$\Rightarrow y = 2x$$

Q. 13. $v \frac{dv}{dx} = \cos^2 x$

$$\Rightarrow v dv = \cos^2 x dx$$

$$\Rightarrow \int_1^v v dv = \int_0^x \cos^2 x dx$$

$$\Rightarrow \frac{v^2}{2} \Big|_1^v = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^x$$

$$\Rightarrow v^2 \Big|_1^v = \left(x + \frac{1}{2} \sin 2x \right) \Big|_0^x$$

$$\Rightarrow v^2 - 1 = x + \frac{1}{2} \sin 2x$$

$$\Rightarrow v^2 = x + \frac{1}{2} \sin 2x + 1$$

$$\Rightarrow v = \sqrt{x + \frac{1}{2} \sin 2x + 1}$$

Q. 14. $\frac{dy}{dx} = y \sin x$

$$\Rightarrow \frac{1}{y} dy = \sin x dx$$

$$\Rightarrow \int_{\sqrt{e}}^y \frac{1}{y} dy = \int_{\frac{\pi}{3}}^x \sin x dx$$

$$\Rightarrow \log_e y \Big|_{\sqrt{e}}^y = -\cos x \Big|_{\frac{\pi}{3}}^x$$

$$\Rightarrow \log_e y - \log_e \sqrt{e} = -\cos x + \cos \frac{\pi}{3}$$

$$\Rightarrow \log_e y - \frac{1}{2} = -\cos x + \frac{1}{2}$$

$$\Rightarrow \log_e y = 1 - \cos x$$

$$\Rightarrow y = e^{1 - \cos x}$$

Q. 15. (i) $\frac{dy}{dx} = y \cos x$

$$\Rightarrow \frac{1}{y} dy = \cos x dx$$

$$\Rightarrow \int_2^y \frac{1}{y} dy = \int_{\frac{\pi}{6}}^x \cos x dx$$

$$\Rightarrow \log_e y \Big|_2^y = \sin x \Big|_{\frac{\pi}{6}}^x$$

$$\Rightarrow \log_e y - \log_e 2 = \sin x - \sin \frac{\pi}{6}$$

$$\Rightarrow \log_e \frac{y}{2} = \sin x - \frac{1}{2}$$

$$\Rightarrow \frac{y}{2} = e^{\sin x - \frac{1}{2}}$$

$$\Rightarrow y = 2e^{\sin x - \frac{1}{2}}$$

(ii) Let $x = \frac{\pi}{2}$

$$\Rightarrow y = 2e^{1 - \frac{1}{2}} = 2e^{\frac{1}{2}} = 2\sqrt{e}$$

Q. 16. $2 \frac{dy}{dx} = xy + x$

$$\Rightarrow 2 \frac{dy}{dx} = x(y + 1)$$

$$\Rightarrow \frac{2}{y+1} dy = x dx$$

$$\Rightarrow \int_2^y \frac{2}{y+1} dy = \int_1^x x dx$$

$$\Rightarrow 2 \log_e |y+1| \Big|_2^y = \frac{x^2}{2} \Big|_1^x$$

$$\Rightarrow 2(\log_e |y+1| - \log_e 3) = \frac{x^2}{2} - \frac{1}{2}$$

$$\Rightarrow 2 \log_e \left| \frac{y+1}{3} \right| = \frac{x^2 - 1}{2}$$

$$\Rightarrow \log_e \left| \frac{y+1}{3} \right| = \frac{x^2 - 1}{4}$$

$$\Rightarrow \frac{y+1}{3} = e^{\frac{x^2 - 1}{4}}$$

$$\begin{aligned} \Rightarrow y + 1 &= 3e^{\frac{x^2-1}{4}} \\ \Rightarrow y &= 3e^{\frac{x^2-1}{4}} - 1 \quad \dots \text{let } x = 2 \\ \Rightarrow y &= 3e^{\frac{3}{4}} - 1 \\ \Rightarrow y &= 5.35 \end{aligned}$$

Q. 17. $x \frac{dy}{dx} = \sqrt{4 - y^2}$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{4 - y^2}} dy &= \frac{1}{x} dx \\ \Rightarrow \int_0^y \frac{1}{\sqrt{4 - y^2}} dy &= \int_1^x \frac{1}{x} dx \\ \Rightarrow \sin^{-1} \frac{y}{2} \Big|_0^y &= \log_e x \Big|_1^x \\ \Rightarrow \sin^{-1} \frac{y}{2} - \sin^{-1} 0 &= \log_e x - \log_e 1 \\ \Rightarrow \sin^{-1} \frac{y}{2} &= \log_e x \\ \Rightarrow \frac{y}{2} &= \sin(\log_e x) \\ \Rightarrow y &= 2 \sin(\log_e x) \\ &\dots \text{general solution} \end{aligned}$$

Let $x = \sqrt[3]{e^{\frac{\pi}{2}}}$

$$\begin{aligned} \Rightarrow y &= 2 \sin\left(\log_e \sqrt[3]{e^{\frac{\pi}{2}}}\right) \\ \Rightarrow y &= 2 \sin\left(\frac{\pi}{6} \log_e e\right) \\ \Rightarrow y &= 2 \sin \frac{\pi}{6} \\ \Rightarrow y &= 2 \left(\frac{1}{2}\right) \\ \Rightarrow y &= 1 \end{aligned}$$

Exercise 12F

Q. 1. $\frac{d^2y}{dx^2} = 2 \frac{dy}{dx}$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= 2v, \text{ where } v = \frac{dy}{dx} \\ \Rightarrow \int_1^v \frac{dv}{v} &= \int_0^x 2 dx \\ \Rightarrow \log_e v \Big|_1^v &= 2x \Big|_0^x \\ \Rightarrow \log_e v - \log_e 1 &= 2x - 2(0) \\ &\dots \log_e 1 = 0 \\ \Rightarrow \log_e v &= 2x \\ \Rightarrow v &= e^{2x} \quad \dots \text{End of Step 1} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= e^{2x} \\ \Rightarrow \int_0^y dy &= \int_0^x e^{2x} dx \\ \Rightarrow y \Big|_0^y &= \frac{1}{2} e^{2x} \Big|_0^x \\ \Rightarrow y - 0 &= \frac{1}{2} [e^{2x} - e^0] \\ \Rightarrow y &= \frac{1}{2} [e^{2x} - 1] \end{aligned}$$

Q. 2. $\frac{d^2y}{(dx)^2} = \frac{dv}{dt} = v^2$, where $v = \frac{dx}{dt}$

$$\begin{aligned} \Rightarrow \int_1^v \frac{dv}{v^2} &= \int_0^t dt \\ \Rightarrow -\frac{1}{v} \Big|_1^v &= t \Big|_0^t \\ \Rightarrow -\frac{1}{v} - (-1) &= t - 0 \\ \Rightarrow 1 - \frac{1}{v} &= t \\ \Rightarrow \frac{1}{v} &= 1 - t \\ \Rightarrow v &= \frac{1}{1 - t} \quad \dots \text{End of Step 1} \\ \Rightarrow \frac{dx}{dt} &= \frac{1}{1 - t} \\ \Rightarrow \int_0^x dx &= \int_0^t \frac{dt}{1 - t} \\ \Rightarrow x \Big|_0^x &= -\log_e (1 - t) \Big|_0^t \\ \Rightarrow x - 0 &= -\log_e (1 - t) - (-\log_e 1) \\ &\dots \log_e 1 = 0 \\ \Rightarrow x &= \log_e (1 - t)^{-1} \\ \Rightarrow x &= \log_e \left(\frac{1}{1 - t}\right) \end{aligned}$$

Q. 3. $\frac{d^2s}{dt^2} = 6$

$$\begin{aligned} \Rightarrow \frac{dv}{dt} &= 6, \text{ where } v = \frac{ds}{dt} \\ \Rightarrow \int_4^v dv &= \int_0^t 6 dt \\ \Rightarrow v \Big|_4^v &= 6t \Big|_0^t \\ \Rightarrow v - 4 &= 6t \\ \Rightarrow v &= 6t + 4 \quad \dots \text{End of Step 1} \\ \Rightarrow \frac{ds}{dt} &= 6t + 4 \\ \Rightarrow \int_0^s ds &= \int_0^t (6t + 4) dt \\ \Rightarrow s \Big|_0^s &= (3t^2 + 4t) \Big|_0^t \\ \Rightarrow s &= 3t^2 + 4t \end{aligned}$$

Q. 4. $\frac{d^2s}{dt^2} = -\left(\frac{ds}{dt}\right)^2$

$$\Rightarrow \frac{dv}{dt} = -v^2, \text{ where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_{\frac{1}{2}}^v \frac{dv}{v^2} = -\int_0^t dt$$

$$\Rightarrow -\frac{1}{v} \Big|_{\frac{1}{2}}^v = -t \Big|_0^t$$

$$\Rightarrow -\frac{1}{v} - (-2) = -t$$

$$\Rightarrow 2 - \frac{1}{v} = -t$$

$$\Rightarrow \frac{1}{v} = t + 2$$

$$\Rightarrow v = \frac{1}{t + 2} \quad \dots \text{End of Step 1}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{t + 2}$$

$$\Rightarrow \int_0^s ds = \int_0^t \frac{dt}{t + 2}$$

$$\Rightarrow s \Big|_0^s = \log_e (t + 2) \Big|_0^t$$

$$\Rightarrow s - 0 = \log_e (t + 2) - \log_e 2$$

$$\Rightarrow s = \log_e \left(\frac{t + 2}{2} \right)$$

Q. 5. $\frac{d^2x}{dt^2} = \left(\frac{dx}{dt}\right)^2 + 1$

$$\Rightarrow \frac{dv}{dt} = v^2 + 1, \text{ where } v = \frac{dx}{dt}$$

$$\Rightarrow \int_0^v \frac{dv}{v^2 + 1} = \int_0^t dt$$

$$\Rightarrow \tan^{-1} v \Big|_0^v = t \Big|_0^t$$

$$\Rightarrow \tan^{-1} v - \tan^{-1} 0 = t - 0 \quad \dots \tan^{-1} 0 = 0$$

$$\Rightarrow \tan^{-1} v = t$$

$$\Rightarrow v = \tan t \quad \dots \text{End of Step 1}$$

$$\Rightarrow \frac{dx}{dt} = \tan t$$

$$\Rightarrow \int_0^x dx = \int_0^t \tan t \, dt$$

$$\Rightarrow x \Big|_0^x = -\log_e |\cos t| \Big|_0^t$$

$$\Rightarrow x - 0 = -\log_e |\cos t| - (-\log_e |\cos 0|)$$

$$\dots \log_e |\cos 0| = \log_e 1 = 0$$

$$\Rightarrow x = -\log_e |\cos t|$$

$$\Rightarrow x = \log_e |\cos t|^{-1}$$

$$\Rightarrow x = \log_e \left| \frac{1}{\cos t} \right| \quad \text{OR } x = \log_e |\sec t|$$

Q. 6. $\frac{d^2x}{dt^2} = 2x \quad \dots \text{let } v = \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = \frac{d^2y}{dx^2}$

$$\Rightarrow \frac{dv}{dx} = 2x$$

$$\Rightarrow \int_1^v dv = \int_0^x 2x \, dx$$

$$\Rightarrow v \Big|_1^v = x^2 \Big|_0^x$$

$$\Rightarrow v - 1 = x^2 - 0$$

$$\Rightarrow v = x^2 + 1 \quad \dots \text{End of Step 1}$$

$$\Rightarrow \frac{dy}{dx} = x^2 + 1$$

$$\Rightarrow \int_{10}^y dy = \int_0^x (x^2 + 1) \, dx$$

$$\Rightarrow y \Big|_{10}^y = \left(\frac{x^3}{3} + x \right) \Big|_0^x$$

$$\Rightarrow y - 10 = \frac{x^3}{3} + x$$

$$\Rightarrow y = \frac{x^3}{3} + x + 10$$

Exercise 12G

Q. 1. $\frac{d^2y}{dx^2} = y$

$$\Rightarrow v \frac{dv}{dy} = y, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_1^v v \, dv = \int_1^y y \, dy$$

$$\Rightarrow \frac{v^2}{2} \Big|_1^v = \frac{y^2}{2} \Big|_1^y$$

$$\Rightarrow \frac{v^2}{2} = \frac{y^2}{2}$$

$$\Rightarrow v^2 = y^2$$

$$\Rightarrow v = y \quad (v = -y \text{ won't work})$$

$$\Rightarrow \frac{dy}{dx} = y$$

$$\Rightarrow \int_1^y \frac{dy}{y} = \int_0^x dx$$

$$\Rightarrow \log_e y \Big|_1^y = x \Big|_0^x$$

$$\Rightarrow \log_e y - \log_e 1 = x - 0 \quad \dots \log_e 1 = 0$$

$$\Rightarrow \log_e y = x$$

$$\Rightarrow y = e^x$$

Q. 2. $\frac{d^2s}{dt^2} = 9s$

$$\Rightarrow v \frac{dv}{ds} = 9s, \text{ where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_{-6}^v v \, dv = \int_2^s 9s \, ds$$

$$\Rightarrow \frac{v^2}{2} \Big|_{-6}^v = \frac{9s^2}{2} \Big|_2^s$$

$$\Rightarrow \frac{v^2}{2} - 18 = \frac{9s^2}{2} - 18$$

$$\Rightarrow v^2 = 9s^2$$

$$\Rightarrow v = -3s$$

($v = 3s$ won't work)

$$\Rightarrow \frac{ds}{dt} = -3s$$

$$\Rightarrow \int_e^s \frac{ds}{s} = \int_0^t -3dt$$

$$\Rightarrow \log_e s \Big|_e^s = -3t \Big|_0^t$$

$$\Rightarrow \log_e s - \log_e e = -3t$$

$$\Rightarrow \log_e s - 1 = -3t$$

$$\Rightarrow \log_e s = 1 - 3t$$

$$\Rightarrow s = e^{1-3t}$$

Q. 3. $\frac{d^2y}{dx^2} = -\frac{2}{y^5}$

$$\Rightarrow v \frac{dv}{dy} = -\frac{2}{y^5}, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_1^v v \, dv = \int_1^y -\frac{2}{y^5} \, dy$$

$$\Rightarrow \frac{v^2}{2} \Big|_1^v = \frac{1}{2y^4} \Big|_1^y$$

$$\Rightarrow \frac{v^2}{2} - \frac{1}{2} = \frac{1}{2y^4} - \frac{1}{2}$$

$$\Rightarrow v^2 = \frac{1}{y^4}$$

$$\Rightarrow v = \frac{1}{y^2} \quad \left(v = -\frac{1}{y^2} \text{ won't work} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y^2}$$

$$\Rightarrow \int_1^y y^2 \, dy = \int_{\frac{1}{3}}^x dx$$

$$\Rightarrow \frac{y^3}{3} \Big|_1^y = x \Big|_{\frac{1}{3}}^x$$

$$\Rightarrow \frac{y^3}{3} - \frac{1}{3} = x - \frac{1}{3}$$

$$\Rightarrow y^3 = 3x$$

$$\Rightarrow y = \sqrt[3]{3x}$$

Q. 4. $\frac{d^2y}{dx^2} = -y$

$$\Rightarrow v \frac{dv}{dy} = -y, \text{ where } v = \frac{dy}{dx}$$

$$\Rightarrow \int_2^v v \, dv = \int_0^y -y \, dy$$

$$\Rightarrow \frac{v^2}{2} \Big|_2^v = -\frac{y^2}{2} \Big|_0^y$$

$$\Rightarrow \frac{v^2}{2} - 2 = -\frac{y^2}{2}$$

$$\Rightarrow v^2 - 4 = -y^2$$

$$\Rightarrow v^2 = 4 - y^2$$

$$\Rightarrow v = \sqrt{4 - y^2}$$

($v = -\sqrt{4 - y^2}$ won't work)

$$\Rightarrow \frac{dy}{dx} = \sqrt{4 - y^2}$$

$$\Rightarrow \int_0^y \frac{dy}{\sqrt{4 - y^2}} = \int_0^x dx$$

$$\Rightarrow \sin^{-1} \frac{y}{2} \Big|_0^y = x \Big|_0^x$$

$$\Rightarrow \sin^{-1} \frac{y}{2} - \sin^{-1} 0 = x - 0$$

... $\sin^{-1} 0 = 0$

$$\Rightarrow \sin^{-1} \frac{y}{2} = x$$

$$\Rightarrow \frac{y}{2} = \sin x$$

$$\Rightarrow y = 2 \sin x$$

Q. 5. $\frac{d^2x}{dt^2} = 2x(9 + x^2)$

$$\Rightarrow v \frac{dv}{dx} = 2x(9 + x^2), \text{ where } v = \frac{dx}{dt}$$

$$\Rightarrow \int_9^v v \, dv = \int_0^x 2x(9 + x^2) \, dx$$

$$\Rightarrow \frac{v^2}{2} \Big|_9^v = \frac{1}{2}(9 + x^2)^2 \Big|_0^x$$

(using the substitution: $u = 9 + x^2$)

$$\Rightarrow \frac{v^2}{2} - \frac{81}{2} = \frac{1}{2}(9 + x^2)^2 - \frac{81}{2}$$

$$\Rightarrow v^2 = (9 + x^2)^2$$

$$\Rightarrow v = 9 + x^2$$

($v = -(9 + x^2)$ won't work)

$$\Rightarrow \frac{dx}{dt} = 9 + x^2$$

$$\Rightarrow \int_3^x \frac{dx}{9 + x^2} = \int_0^t dt$$

$$\Rightarrow \frac{1}{3} \tan^{-1} \frac{x}{3} \Big|_3^x = t \Big|_0^t$$

$$\begin{aligned} \Rightarrow \frac{1}{3}(\tan^{-1} \frac{x}{3} - \tan^{-1} 1) &= t - 0 \\ \Rightarrow \frac{1}{3}(\tan^{-1} \frac{x}{3} - \frac{\pi}{4}) &= t \\ \Rightarrow \tan^{-1} \frac{x}{3} - \frac{\pi}{4} &= 3t \\ \Rightarrow \tan^{-1} \frac{x}{3} &= 3t + \frac{\pi}{4} \\ \Rightarrow \frac{x}{3} &= \tan(3t + \frac{\pi}{4}) \\ \Rightarrow x &= 3 \tan(3t + \frac{\pi}{4}) \end{aligned}$$

Q. 6. $\frac{d^2x}{dt^2} = \frac{3x^2}{2}$

$$\begin{aligned} \Rightarrow v \frac{dv}{dx} &= \frac{3x^2}{2}, \text{ where } v = \frac{dx}{dt} \\ \Rightarrow \int_{-8}^v 2v \, dv &= \int_4^x 3x^2 \, dx \\ \Rightarrow v^2 \Big|_{-8}^v &= x^3 \Big|_4^x \\ \Rightarrow v^2 - 64 &= x^3 - 64 \\ \Rightarrow v^2 &= x^3 \\ \Rightarrow v &= -\sqrt{x^3} \\ &\quad (v = \sqrt{x^3} \text{ won't work}) \\ \Rightarrow \frac{dx}{dt} &= -x^{\frac{3}{2}} \\ \Rightarrow \int_4^x x^{-\frac{3}{2}} \, dx &= \int_0^t -dt \\ \Rightarrow \frac{-2}{\sqrt{x}} \Big|_4^x &= -t \Big|_0^t \\ \Rightarrow \frac{-2}{\sqrt{x}} - (-1) &= -t \\ \Rightarrow \frac{2}{\sqrt{x}} &= t + 1 \\ \Rightarrow \frac{\sqrt{x}}{2} &= \frac{1}{t + 1} \\ \Rightarrow \sqrt{x} &= \frac{2}{t + 1} \\ \Rightarrow x &= \left(\frac{2}{t + 1}\right)^2 \end{aligned}$$

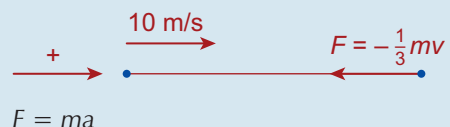
Q. 7. $\frac{d^2y}{dx^2} + \frac{2}{y^3} = 0$

$$\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= -2y^{-3} \\ \text{Let } \frac{dy}{dx} &= v \Rightarrow \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v \frac{dv}{dy} \\ \Rightarrow v \frac{dv}{dy} &= -2y^{-3} \\ \Rightarrow \int_{\sqrt{2}}^v v \, dv &= \int_1^y -2y^{-3} \, dy \\ \Rightarrow \frac{v^2}{2} \Big|_{\sqrt{2}}^v &= \frac{1}{y^2} \Big|_1^y \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{v^2}{2} - 1 &= \frac{1}{y^2} - 1 \\ \Rightarrow v^2 &= \frac{2}{y^2} \\ \Rightarrow v &= \frac{\sqrt{2}}{y} \quad (v = -\frac{\sqrt{2}}{y} \text{ won't work}) \\ \Rightarrow \frac{dy}{dx} &= \frac{\sqrt{2}}{y} \\ \Rightarrow \int_1^y y \, dy &= \int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2} \, dx \\ \Rightarrow \frac{y^2}{2} \Big|_1^y &= \sqrt{2} x \Big|_{\sqrt{2}}^{\sqrt{18}} \\ \Rightarrow \frac{y^2}{2} - \frac{1}{2} &= 6 - 2 \\ \Rightarrow y^2 - 1 &= 8 \\ \Rightarrow y^2 &= 9 \\ \Rightarrow y &= \pm 3 \end{aligned}$$

Exercise 12H

Q. 1. (i)



$$\begin{aligned} \Rightarrow -\frac{1}{3}mv &= ma \\ \Rightarrow a &= -\frac{1}{3}v \\ \Rightarrow \frac{dv}{dt} &= -\frac{1}{3}v \\ \Rightarrow \int_{10}^v \frac{dv}{v} &= \int_0^t -\frac{1}{3} \, dt \\ \Rightarrow \log_e v \Big|_{10}^v &= -\frac{1}{3}t \Big|_0^t \\ \Rightarrow \log_e v - \log_e 10 &= -\frac{1}{3}t \\ \Rightarrow \log_e \frac{v}{10} &= -\frac{1}{3}t \\ \Rightarrow \frac{v}{10} &= e^{-\frac{1}{3}t} \\ \Rightarrow v &= 10e^{-\frac{1}{3}t} \end{aligned}$$

...Equation 1

When $t = 3$, $v = 10e^{-1} = \frac{10}{e}$ m/s

(ii) $\frac{ds}{dt} = 10e^{-\frac{1}{3}t}$...from Equation 1

$$\begin{aligned} \Rightarrow \int_0^s ds &= \int_0^t 10e^{-\frac{1}{3}t} \, dt \\ \Rightarrow s \Big|_0^s &= -30e^{-\frac{1}{3}t} \Big|_0^t \end{aligned}$$

$$\Rightarrow s - 0 = -30e^{-\frac{1}{3}t} - (-30)$$

$$\Rightarrow s = 30(1 - e^{-\frac{1}{3}t})$$

When $t = 3$

$$s = 30(1 - e^{-1}) = 30\left(1 - \frac{1}{e}\right)$$

(iii) As $t \rightarrow \infty$, $e^{-\frac{1}{3}t} \rightarrow 0$

$$\therefore s \rightarrow 30(1 - 0) = 30 \text{ m}$$

Q. 2. $a = 25v + v^3$

$$\Rightarrow v \frac{dv}{ds} = v(25 + v^2)$$

$$\Rightarrow dv = (25 + v^2) ds \quad \text{QED}$$

$$\Rightarrow \int_0^v \frac{dv}{25 + v^2} = \int_0^{0.01} ds$$

$$\Rightarrow \frac{1}{5} \left[\tan^{-1} \frac{v}{5} \right]_0^v = [s]_0^{0.01}$$

$$\Rightarrow \tan^{-1} \frac{v}{5} = 0.05$$

$$\Rightarrow \frac{v}{5} = \tan 0.05$$

$$\Rightarrow v = 5 \tan 0.05$$

$$\Rightarrow v = 0.25 \text{ m/s}$$

Q. 3. $a = v^2 + 100$

$$\Rightarrow \frac{dv}{dt} = v^2 + 100$$

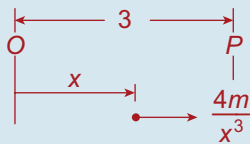
$$\Rightarrow \int_0^{20} \frac{dv}{v^2 + 100} = \int_0^t dt$$

$$\Rightarrow \frac{1}{10} \left[\tan^{-1} \frac{v}{10} \right]_0^{20} = t$$

$$\Rightarrow t = \frac{1}{10} [\tan^{-1} 2]$$

$$\Rightarrow t = 0.11 \text{ s}$$

Q. 4. Forces



NZL: $\Sigma F = ma$

$$\rightarrow \frac{4m}{x^3} = m v \frac{dv}{dx}$$

(i) $\Rightarrow v dv = 4x^{-3} dx \quad \text{QED}$

$$\Rightarrow \int_{-\frac{\sqrt{5}}{3}}^v v dv = 4 \int_3^x x^{-3} dx$$

$$\Rightarrow \frac{1}{2} \left[v^2 \right]_{-\frac{\sqrt{5}}{3}}^v = -2 \left[\frac{1}{x^2} \right]_3^x$$

$$\Rightarrow v^2 - \frac{5}{9} = 4 \left[\frac{1}{9} - \frac{1}{x^2} \right]$$

$$\Rightarrow v^2 = 1 - \frac{4}{x^2}$$

$$\Rightarrow v = \sqrt{1 - \frac{4}{x^2}}$$

(ii) For $v = 0$, $\frac{4}{x^2} = 1 \Rightarrow x = 2 \text{ m}$

(iii) When $x = \frac{5}{2}$

$$v = \sqrt{1 - \frac{4}{\left(\frac{25}{4}\right)}}$$

$$= \sqrt{1 - \frac{16}{25}}$$

$$= \sqrt{\frac{9}{25}}$$

$$\Rightarrow v = \frac{3}{5} \text{ m/s}$$

Q. 5. (i) $F = ma$

$$\Rightarrow 8a = (40 - 3\sqrt{x})$$

$$\Rightarrow a = \frac{40 - 3\sqrt{x}}{8}$$

$$\Rightarrow v \frac{dv}{dx} = \frac{40 - 3\sqrt{x}}{8}$$

$$\Rightarrow \int_0^v 8v dv = \int_0^x (40 - 3\sqrt{x}) dx$$

$$\Rightarrow 4v^2 \Big|_0^v = (40x - 2\sqrt{x^3}) \Big|_0^x$$

$$\Rightarrow 4v^2 = 40x - 2\sqrt{x^3}$$

$$\Rightarrow v^2 = 10x - \frac{1}{2}\sqrt{x^3}$$

$$\Rightarrow v = \sqrt{10x - \frac{1}{2}\sqrt{x^3}} \quad \dots \text{let } x = 100$$

$$\Rightarrow v = \sqrt{1000 - \frac{1}{2}(1000)}$$

$$= \sqrt{500}$$

$$= 22.36 \text{ m/s}$$

(ii) $v^2 = 10x - \frac{1}{2}\sqrt{x^3} \quad \dots \text{let } v = 0$

$$\Rightarrow 10x - \frac{1}{2}\sqrt{x^3} = 0$$

$$\Rightarrow 20x - \sqrt{x^3} = 0$$

$$\Rightarrow 20x - x\sqrt{x} = 0$$

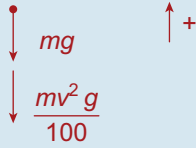
$$\Rightarrow x(20 - \sqrt{x}) = 0$$

$$\Rightarrow 20 - \sqrt{x} = 0$$

$$\Rightarrow \sqrt{x} = 20$$

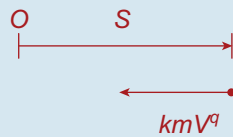
$$\Rightarrow x = 400 \text{ m}$$

Q. 6. Forces



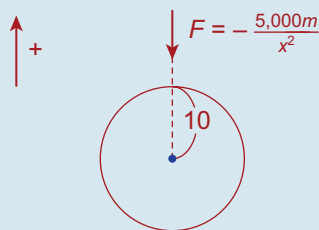
$$\begin{aligned} \text{NZL: } \Sigma F &= ma \\ + \uparrow - mg \left[1 + \frac{v^2}{100} \right] &= m \frac{dv}{dt} \\ \Rightarrow -g \left[\frac{100 + v^2}{100} \right] &= \frac{dv}{dt} \\ \Rightarrow \int_{120}^0 \frac{dv}{v^2 + 100} &= -\frac{g}{100} \int_0^t dt \\ \Rightarrow \frac{1}{10} \left[\tan^{-1} \frac{v}{10} \right]_{120}^0 &= -\frac{g}{100} t \\ \Rightarrow \frac{gt}{10} &= \tan^{-1} 12 \\ \Rightarrow t &= 1.518 \\ \Rightarrow t &= 1.5 \text{ s} \end{aligned}$$

Q. 7. Forces



$$\begin{aligned} \text{NZL: } \Sigma F &= ma \\ \rightarrow -kmv^q &= mv \frac{dv}{ds} \\ \Rightarrow \int_u^0 v^{1-q} dv &= -k \int_0^s ds \\ \Rightarrow \frac{1}{2-q} \left[v^{2-q} \right]_u^0 &= -ks \\ \Rightarrow \frac{1}{2-q} [0^{2-q} - u^{2-q}] &= -ks \\ \Rightarrow ks &= \frac{u^{2-q}}{2-q} \\ \Rightarrow s &= \frac{u^{2-q}}{(2-q)^k} \quad \text{QED} \end{aligned}$$

Q. 8.



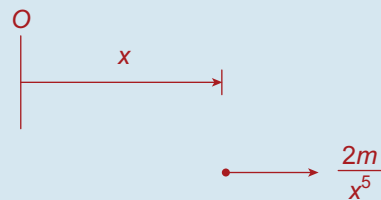
$$\begin{aligned} \text{(i) } F &= ma \\ \Rightarrow -\frac{5,000 m}{x^2} &= ma \\ \Rightarrow a &= -\frac{5,000}{x^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow v \frac{dv}{dx} &= -\frac{5,000}{x^2} \\ \Rightarrow \int_{20}^v v dv &= \int_{10}^x -\frac{5,000}{x^2} dx \\ \Rightarrow \frac{v^2}{2} \Big|_{20}^v &= \frac{5,000}{x} \Big|_{10}^x \\ \Rightarrow \frac{v^2}{2} - 200 &= \frac{5,000}{x} - 500 \\ \Rightarrow \frac{v^2}{2} &= \frac{5,000}{x} - 300 \\ \Rightarrow v^2 &= \frac{10,000}{x} - 600 \\ \Rightarrow v &= \sqrt{\frac{10,000}{x} - 600} \\ \dots \text{let } x &= \frac{30}{7} + 10 = \frac{100}{7} \\ \Rightarrow v &= \sqrt{700 - 600} \\ &= \sqrt{100} \\ &= 10 \text{ m/s} \end{aligned}$$

(ii) At its greatest height, $v = 0$

$$\begin{aligned} \Rightarrow \frac{10,000}{x} - 600 &= 0 \\ \Rightarrow 10,000 - 600x &= 0 \\ \Rightarrow x &= \frac{10,000}{600} \\ &= \frac{50}{3} \text{ m} \\ \therefore \text{height above surface} &= \frac{50}{3} - 10 \\ &= \frac{20}{3} \text{ m} \end{aligned}$$

Q. 9. Forces



$$\begin{aligned} \text{NZL: } \Sigma F &= ma \\ + \frac{2m}{x^5} &= mv \frac{dv}{dx} \\ \Rightarrow \int_0^v v dv &= 2 \int_d^x x^{-5} dx \\ \Rightarrow \frac{1}{2} [v^2]_0^v &= -\frac{2}{4} \left[\frac{1}{x^4} \right]_d^x \\ \Rightarrow v^2 &= \left[\frac{1}{d^4} - \frac{1}{x^4} \right] \\ \Rightarrow v &= \sqrt{\frac{1}{d^4} - \frac{1}{x^4}} \quad \dots \textcircled{1} \end{aligned}$$

(i) When $x = \sqrt{3}d$,

$$v = \sqrt{\frac{1}{d^4} - \frac{1}{9d^4}}$$

$$\Rightarrow v = \sqrt{\frac{9-1}{9d^4}}$$

$$\Rightarrow v = \frac{2\sqrt{2}}{3d^2} \quad \text{QED}$$

(ii) From (1), $v_T = \lim_{x \rightarrow \infty} \sqrt{\frac{1}{d^4} - \frac{1}{x^4}}$
where $v_T =$ Terminal Velocity

$$\Rightarrow v_T = \frac{1}{d^2} \quad \text{QED}$$

Q. 10. $\frac{d^2s}{dt^2} = a$, Let $\frac{ds}{dt} = v$

$$\Rightarrow \frac{dv}{dt} = a$$

$$\Rightarrow \int_u^v dv = a \int_0^t dt$$

$$\Rightarrow v - u = at$$

$$\Rightarrow v = u + at$$

$$\Rightarrow \frac{ds}{dt} = u + at$$

$$\Rightarrow \int_0^s ds = \int_0^t (u + at) dt$$

$$\Rightarrow s = ut + \frac{at^2}{2} \quad \text{QED}$$

Q. 11. $\frac{d^2s}{dt^2} = a$

$$\Rightarrow v \frac{dv}{ds} = a$$

$$\Rightarrow \int_u^v v dv = a \int_0^s ds$$

$$\Rightarrow \frac{1}{2} [v^2]_u^v = as$$

$$\Rightarrow v^2 - u^2 = 2as$$

$$\Rightarrow v^2 = u^2 + 2as \quad \text{QED}$$

Exercise 12I

Q. 1. $\frac{dv}{dt} = v^2 + 4$

Note: the integration does require a substitution.

$$\Rightarrow \int_2^v \frac{dv}{v^2 + 4} = \int_0^t dt$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{v}{2} \Big|_2^v = t \Big|_0^t$$

$$\Rightarrow \frac{1}{2} [\tan^{-1} \frac{v}{2} - \tan^{-1} 1] = t - 0$$

$$\dots \tan^{-1} 1 = \frac{\pi}{4}$$

$$\Rightarrow t = \frac{1}{2} [\tan^{-1} \frac{v}{2} - \frac{\pi}{4}] \quad \dots \text{let } v = 6$$

$$\Rightarrow t = \frac{1}{2} [\tan^{-1} 3 - \frac{\pi}{4}]$$

$$= 0.23 \text{ s}$$

Q. 2. $\frac{d^2s}{dt^2} = -\left(\frac{ds}{dt}\right)^2$

$$\Rightarrow \frac{dv}{dt} = -v^2 \quad \text{where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_1^v \frac{dv}{v^2} = -\int_0^t dt$$

$$\Rightarrow -\frac{1}{v} \Big|_1^v = -t \Big|_0^t$$

$$\Rightarrow \frac{1}{v} \Big|_1^v = t \Big|_0^t$$

$$\Rightarrow \frac{1}{v} - 1 = t - 0$$

$$\Rightarrow \frac{1}{v} = t + 1$$

$$\Rightarrow v = \frac{1}{t + 1}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{t + 1}$$

$$\Rightarrow \int_0^s ds = \int_0^1 \frac{dt}{t + 1}$$

$$\Rightarrow s \Big|_0^s = \log_e (t + 1) \Big|_0^1$$

$$\Rightarrow s - 0 = \log_e 2 - \log_e 1 \quad \dots \log_e 1 = 0$$

$$\Rightarrow s = \log_e 2$$

Q. 3. $\frac{d^2s}{dt^2} = \frac{1}{2} \left(\frac{ds}{dt}\right)$

$$\Rightarrow \frac{dv}{dt} = \frac{1}{2}v \quad \text{where } v = \frac{ds}{dt}$$

$$\Rightarrow \int_3^v \frac{dv}{v} = \int_0^t \frac{1}{2} dt$$

$$\Rightarrow \log_e v \Big|_3^v = \frac{1}{2} t \Big|_0^t$$

$$\Rightarrow \log_e v - \log_e 3 = \frac{1}{2} t$$

$$\Rightarrow \log_e \frac{v}{3} = \frac{1}{2}t$$

$$\Rightarrow \frac{v}{3} = e^{\frac{1}{2}t}$$

$$\Rightarrow v = 3e^{\frac{1}{2}t}$$

$$\Rightarrow \frac{ds}{dt} = 3e^{\frac{1}{2}t}$$

$$\Rightarrow \int_0^s ds = \int_4^5 3e^{\frac{1}{2}t} dt$$

(since the 5th second is from $t = 4$ to $t = 5$)

$$\Rightarrow s \Big|_0^s = 6e^{\frac{1}{2}t} \Big|_4^5$$

$$\Rightarrow s = 6(e^{\frac{5}{2}} - e^2) \text{ m}$$

Q. 4. $\frac{d^2s}{dt^2} = k \left(\frac{ds}{dt}\right)^2 \Rightarrow \frac{dv}{dt} = -kv^2$

when $t = 0, s = 0, v = 20$;

when $s = 100, v = 10$.

$$\Rightarrow \int_{20}^v v^{-2} dv = \int_0^t -k dt$$

$$\Rightarrow -\frac{1}{v} \Big|_{20}^v = -kt \Big|_0^t$$

$$\Rightarrow \frac{1}{v} \Big|_{20}^v = kt \Big|_0^t$$

$$\Rightarrow \frac{1}{v} - \frac{1}{20} = kt$$

$$\Rightarrow \frac{1}{v} = kt + \frac{1}{20}$$

$$\Rightarrow \frac{1}{v} = \frac{20kt + 1}{20}$$

$$\Rightarrow v = \frac{20}{20kt + 1}$$

$$\Rightarrow \frac{ds}{dt} = \frac{20}{20kt + 1}$$

$$\Rightarrow \int_0^{100} ds = \int_0^t \frac{20}{20kt + 1}$$

$$\Rightarrow s \Big|_0^{100} = \frac{1}{k} \log_e (20kt + 1) \Big|_0^t$$

$$\Rightarrow \frac{1}{k} [\log_e (20kt + 1) - \log_e 1] = 100 - 0$$

... $\log_e 1 = 0$

$$\Rightarrow \frac{1}{k} \log_e (20kt + 1) = 100$$

Equation 1

...when $\frac{20}{20kt + 1} = 10$

Equation 2

$$\Rightarrow \underbrace{20kt + 1 = 2}$$

Equation 3

$$\Rightarrow \frac{1}{k} \log_e 2 = 100$$

$$\Rightarrow k = \frac{1}{100} \log_e 2$$

Putting this into equation 3 gives

$$\frac{1}{5} (\log_e 2)t + 1 = 2$$

$$\Rightarrow t = \frac{5}{\log_e 2} = 7.2 \text{ s}$$

Q. 5. $a = -\frac{v^3}{25} \dots \textcircled{1}$

$$\Rightarrow v \frac{dv}{dx} = -\frac{v^3}{25}$$

$$\Rightarrow \int_{25}^v \frac{dv}{v^2} = -\frac{1}{25} \int_0^x dx$$

$$\Rightarrow -\left[\frac{1}{v}\right]_{25}^v = -\frac{1}{25} \left[x\right]_0^x$$

$$\Rightarrow \frac{1}{v} - \frac{1}{25} = \frac{x}{25}$$

$$\Rightarrow \frac{1}{v} = \frac{x+1}{25}$$

(i) $\Rightarrow v = \frac{25}{x+1}$

$$\Rightarrow v \Big|_{x=99} = \frac{1}{4} \text{ m/s}$$

(ii) $\frac{dv}{dt} = -\frac{v^3}{25}$ from $\textcircled{1}$

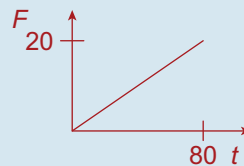
$$\Rightarrow \int_0^5 v^{-3} dv = -\frac{1}{25} \int_0^t dt$$

$$\Rightarrow -\frac{1}{2} \left[\frac{1}{v^2}\right]_{10}^5 = -\frac{1}{25} t$$

$$\Rightarrow t = \frac{25}{2} \left[\frac{1}{25} - \frac{1}{100}\right]$$

$$\Rightarrow t = \frac{3}{8} \text{ s}$$

Q. 6.



$$F = \frac{20}{80}t + 0$$

$$\Rightarrow F = \frac{t}{4}$$

$$\boxed{2} \rightarrow \frac{t}{4}$$

NZL: $\Sigma F = ma$

$$+ \frac{t}{4} = 2a$$

(i) $\Rightarrow a = \frac{t}{8}$ **QED**

(ii) $\frac{dv}{dt} = \frac{t}{8}$

$$\Rightarrow \int_0^v dv = \frac{1}{8} \int_0^t t dt$$

$$\Rightarrow v = \frac{t^2}{16} \dots \textcircled{1}$$

$$\Rightarrow \frac{dx}{dt} = \frac{t^2}{16}$$

$$\Rightarrow \int_0^x dx = \frac{1}{16} \int_0^t t^2 dt$$

$$\Rightarrow x = \frac{t^3}{48} \dots \textcircled{2}$$

From $\textcircled{1}$ $t = 4v^{\frac{1}{2}}$

From $\textcircled{2}$ $x = \left(\frac{4v^{\frac{1}{2}}}{48}\right)^3$

$$\Rightarrow x = \frac{64v^{\frac{3}{2}}}{48}$$

$$\Rightarrow x = \frac{4v^{\frac{3}{2}}}{3}$$

$$\Rightarrow 3x = 4v^{\frac{3}{2}}v$$

$$9x^2 = 16v^3$$

Exercise 12J

Q. 1.



$$P = Fv$$

$$\Rightarrow F = \frac{P}{v}$$

NZL: $\Sigma F = ma$

$$+ \frac{25m}{v} = m \frac{dv}{dt}$$

$$\Rightarrow \int_1^3 v dv = 25 \int_0^t dt$$

$$\Rightarrow 25t = \frac{1}{2}[v^2]_1^3$$

$$\Rightarrow t = \frac{1}{50}[9 - 1]$$

$$\Rightarrow t = \frac{4}{25}$$

$$\Rightarrow t = 0.16$$

Q. 2. $F = \frac{P}{v} = \frac{25km u_0^2}{v}$



NZL: $\Sigma F = ma$

$$+ \frac{25 km u_0^2}{v} = mv \frac{dv}{ds}$$

(i) $\Rightarrow v^2 dv = 25k u_0^2 ds$ QED

(ii) $\int_{u_0}^{4u_0} v^2 dv = 25k u_0^2 \int_0^s ds$

$$\Rightarrow \frac{1}{3}[v^3]_{u_0}^{4u_0} = 25k u_0^2 s$$

$$\Rightarrow \frac{1}{3}[64u_0^3 - u_0^3] = 25k u_0^2 s$$

$$\Rightarrow 21u_0^3 = 25k u_0^2 s$$

$$\Rightarrow s = \frac{21u_0}{25k} \text{ QED}$$

Q. 3. $F = \frac{P}{v} = \frac{75,000}{v}$



NZL: $\Sigma F = ma$

$$\rightarrow \frac{75,000}{v} = 1,000a$$

(i) $\Rightarrow a = \frac{75}{v} \dots \textcircled{1}$

$$\Rightarrow \frac{dv}{dt} = \frac{75}{v}$$

(ii) $\Rightarrow \int_5^{25} v dv = 75 \int_0^t dt$

$$\Rightarrow \frac{1}{2}[v^2]_5^{25} = 75t$$

$$\Rightarrow \frac{1}{2}[625 - 25] = 75t$$

$$\Rightarrow t = \frac{600}{150}$$

$$\Rightarrow t = 4 \text{ s}$$

(iii) From $\textcircled{1}$

$$v \frac{dv}{dx} = \frac{75}{v}$$

$$\Rightarrow \int_5^{25} v^2 dv = 75 \int_0^x dx$$

$$\Rightarrow \frac{1}{3}[v^3]_5^{25} = 75x$$

$$\Rightarrow x = \frac{1}{225}[25^3 - 5^3]$$

$$\Rightarrow x = \frac{620}{9}$$

$$\Rightarrow x = 68.89 \text{ m}$$